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A basic class of symmetric orthogonal functions using the extended Sturm–Liouville theorem for symmetric functions

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Abstract

By using the extended Sturm–Liouville theorem for symmetric functions, we introduced a basic class of symmetric orthogonal polynomials (BCSOP) in a previous paper. The mentioned class satisfies a differential equation of the form

$$x^2(px^2+q)\Phi_n''(x)+x(rx^2+s)\Phi_n'(x)-(n(r+(n-1)p)x^2+(1-(-1)^n)s/2)\Phi_n(x)=0$$

and contains four main sequences of symmetric orthogonal polynomials. In this paper, again by using the mentioned theorem, we introduce a basic class of symmetric orthogonal functions (BCSOF) as a generalization of BCSOP and obtain its standard properties. We show that the latter class satisfies the equation

$$x^2(px^2+q)\Phi_n''(x)+x(rx^2+s)\Phi_n'(x)-(\alpha_n x^2+(1-(-1)^n)\beta/2)\Phi_n(x)=0,$$

in which β is a free parameter and $-\alpha_n$ denotes eigenvalues corresponding to BCSOF. We then consider four sub-classes of defined orthogonal functions class and study their properties in detail. Since BCSOF is a generalization of BCSOP for $\beta=s$, the four mentioned sub-classes respectively generalize the generalized ultraspherical polynomials, generalized Hermite polynomials and two other finite sequences of symmetric polynomials, which were introduced in the previous work.

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1. Introduction

Many sequences of special functions are solutions of a usual Sturm–Liouville problem [3,9] and satisfy an orthogonality property like

$$\int_a^b W(x)\Phi_n(x)\Phi_m(x)\,\mathrm{d}x=\left(\int_a^b W(x)\Phi_n^2(x)\,\mathrm{d}x\right)\delta_{n,m}, \tag{1}$$

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in which

$$\delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

and $W(x)$ denotes the weight function. These functions have a symmetry property if

$$\Phi_n(x) = (-1)^n \Phi_n(-x) \quad \text{for all } n \in \mathbf{Z}^+. \tag{2}$$

Recently in [8] we have presented a key theorem by which one can generalize usual Sturm–Liouville problems with symmetric solutions. For instance, by this theorem a basic class of symmetric orthogonal polynomials (BCSOP) is introduced in [7]. Since the main properties of BCSOP and also the extended Sturm–Liouville theorem for symmetric functions are needed in this article, we restate them here.

Theorem 1.1 (*Masjed-Jamei [8]*). *Let $\Phi_n(x) = (-1)^n \Phi_n(-x)$ be a sequence of symmetric functions that satisfies the differential equation*

$$A(x)\Phi_n''(x) + B(x)\Phi_n'(x) + (\lambda_n C(x) + D(x) + (1-(-1)^n)E(x)/2)\Phi_n(x) = 0, \tag{3}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$ are real functions and $\{\lambda_n\}$ is a sequence of constants. If the functions $A(x)$, $C(x) > 0$, $D(x)$ and $E(x)$ are even and $B(x)$ is odd then

$$\int_{-\alpha}^{\alpha} W^*(x)\Phi_n(x)\Phi_m(x)\,\mathrm{d}x = \left(\int_{-\alpha}^{\alpha} W^*(x)\Phi_n^2(x)\,\mathrm{d}x\right)\delta_{n,m}, \tag{4}$$

where

$$W^*(x) = C(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}\,\mathrm{d}x\right) = \frac{C(x)}{A(x)}\exp\left(\int \frac{B(x)}{A(x)}\,\mathrm{d}x\right). \tag{4.1}$$

Of course, the weight function defined in (4.1) *must be positive and even on* $[-\alpha, \alpha]$ *and $x = \alpha$ must be a root of the function*

$$A(x)K(x) = A(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}\,\mathrm{d}x\right) = \exp\left(\int \frac{B(x)}{A(x)}\,\mathrm{d}x\right), \tag{4.2}$$

i.e., $A(\alpha)K(\alpha) = 0$. In this sense, since $K(x) = W^*(x)/C(x)$ is an even function, $A(-\alpha)K(-\alpha) = 0$ automatically.

1.1. Generation of BCSOP using Theorem 1.1

Suppose in the generic equation (3)

$$A(x) = x^2(px^2+q), \quad B(x) = x(rx^2+s), \quad C(x) = x^2 > 0, \quad D(x) = 0, \quad E(x) = -s, \tag{5}$$

and $\lambda_n = -n(r+(n-1)p)$, then the following differential equation appears:

$$x^2(px^2+q)\Phi_n''(x) + x(rx^2+s)\Phi_n'(x) - (n(r+(n-1)p)x^2 + (1-(-1)^n)s/2)\Phi_n(x) = 0. \tag{6}$$

According to [7], a basic solution of (6) is a symmetric polynomial class in the form

$$S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{(2i+(-1)^{n+1}+2[n/2])p+r}{(2i+(-1)^{n+1}+2)q+s}\right) x^{n-2k}, \tag{7}$$

in which $\prod_{r=0}^{-1} a_r = 1$. Moreover, the monic form of polynomials (7) satisfies the recurrence relation

$$\bar{S}_{n+1}(x) = x\bar{S}_n(x) + C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix}\bar{S}_{n-1}(x); \quad \bar{S}_0(x) = 1, \quad \bar{S}_1(x) = x, \quad n \in \mathbf{N}, \tag{8}$$

where

$$C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix} = \frac{pqn^2 + ((r-2p)q - (-1)^n ps)n + (r-2p)s(1-(-1)^n)/2}{(2pn+r-p)(2pn+r-3p)}, \tag{8.1}$$

and

$$\overline{S}_n(x) = \overline{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \prod_{i=0}^{[n/2]-1} \frac{(2i+(-1)^{n+1}+2)q+s}{(2i+(-1)^{n+1}+2[n/2])p+r} S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right). \tag{8.2}$$

Since (8) is now explicitly known, to determine the norm square value of the polynomials one can use Favard's theorem [4] by noting that there is orthogonality with respect to a weight function. Hence, the generic form of the orthogonality relation of BCSOP can be designed as

$$\begin{aligned} &\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \overline{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \overline{S}_m\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \mathrm{d}x \\ &\quad = \left((-1)^n \prod_{i=1}^{n} C_i\begin{pmatrix} r & s \\ p & q \end{pmatrix} \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \mathrm{d}x\right)\delta_{n,m}, \end{aligned} \tag{9}$$

where the weight function, by referring to relations (4.1) and (5), is defined as

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^2 \exp\left(\int \frac{(r-4p)x^2+(s-2q)}{x(px^2+q)}\,\mathrm{d}x\right) = \exp\left(\int \frac{(r-2p)x^2+s}{x(px^2+q)}\,\mathrm{d}x\right), \tag{10}$$

and α takes standard values $1, \infty$. Also, by noting the mentioned conditions in Theorem 1.1, the function $(px^2+q)W(p,q,r,s;x)$ must vanish at $x=\alpha$ in order to be valid for the orthogonality property (9). A straightforward result from (7) and (8.2) is that

$$\overline{S}_{2n+1}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x\overline{S}_{2n}\left(\begin{matrix} r+2p & s+2q \\ p & q \end{matrix}\middle| x\right). \tag{11}$$

Relation (11) means that the symmetric sequence $\Phi_n(x)$ defined by

$$\Phi_{2n}(x) = \overline{S}_{2n}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right), \quad \Phi_{2n+1}(x) = x\overline{S}_{2n}\left(\begin{matrix} r+2p & s+2q \\ p & q \end{matrix}\middle| x\right) \tag{12}$$

is a basic solution of generalized Sturm–Liouville equation (6).

On the other hand, by referring to general differential equation (3) and considering the special case (6), one might ask: what would happen if $E(x)$ in (5) were $-\beta$ (a free parameter) instead of $-s$ while both of them are even functions? In this paper, by recalling previous themes, we will answer this question in detail and show that the solution of new equation is a basic class of symmetric orthogonal function (BCSOF), which generalizes BCSOP.

2. Generation of BCSOF using Theorem 1.1

Let us consider the following options for the generic differential equation (3):

$$\begin{aligned} &A(x) = x^2(px^2+q) \text{ even}; \quad B(x) = x(rx^2+s) \text{ odd}; \quad C(x) = x^2 > 0 \text{ even}; \\ &D(x) = 0 \text{ even}; \quad E(x) = -\beta \in \mathbf{R} \text{ even}, \end{aligned} \tag{13}$$

where p, q, r, s and β are free parameters. Therefore, we deal with the equation

$$x^2(px^2+q)\Phi_n''(x) + x(rx^2+s)\Phi_n'(x) - (\alpha_n x^2 + (1-(-1)^n)\beta/2)\Phi_n(x) = 0, \tag{14}$$

where $\lambda_n = -\alpha_n$ is the corresponding eigenvalue to be derived. As the feature of this equation shows, there is a small difference between it and Eq. (6). However, we must solve it separately. Since (14) is independent of β when

$n=2m$; $m \in \mathbf{Z}^+$, without loss of generality we can assume its solution is almost similar to (12) as

$$\Phi_{2n}(x) = \overline{S}_{2n}\left(\begin{array}{cc|c} r & s & \\ p & q & x \end{array}\right), \quad \Phi_{2n+1}(x) = x^{\theta}\overline{S}_{2n}\left(\begin{array}{cc|c} r^* & s^* & \\ p^* & q^* & x \end{array}\right), \quad \theta \in \mathbf{R} - \{0\} \tag{15}$$

and try to obtain the parameters p^*, q^*, r^*, s^* and θ in terms of the five known parameters p, q, r, s and β to somehow get to Eq. (14) and determine the eigenvalues $-\alpha_n$ too. In this way, we should note that the condition $(-1)^{\theta} = -1$ is, however, necessary in the second definition of (15) because $\Phi_n(x)$ must be a symmetric sequence and therefore $\Phi_{2n+1}(-x) = -\Phi_{2n+1}(x)$. To solve the foresaid problem, there is an interesting technique. First, as Eq. (6) and its generic solution shows, if $n=2m$ then

$$x(px^2+q)\Phi''_{2m}(x) + (rx^2+s)\Phi'_{2m}(x) - 2m(r+(2m-1)p)x\Phi_{2m}(x) = 0 \tag{16}$$

has the general solution

$$\Phi_{2m}(x) = S_{2m}\left(\begin{array}{cc|c} r & s & \\ p & q & x \end{array}\right) = \sum_{k=0}^{m}\binom{m}{k}\left(\prod_{j=0}^{m-(k+1)} \frac{(2j-1+2m)p+r}{(2j+1)q+s}\right)x^{2m-2k}. \tag{17}$$

Hence, the second equality of (15) must obey the equation

$$\begin{aligned} &x(p^*x^2+q^*)\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^{-\theta}\Phi_{2m+1}(x)) + (r^*x^2+s^*)\frac{\mathrm{d}}{\mathrm{d}x}(x^{-\theta}\Phi_{2m+1}(x)) \\ &\quad - 2m(r^*+(2m-1)p^*)x^{1-\theta}\Phi_{2m+1}(x) = 0. \end{aligned} \tag{18}$$

After some calculations, (18) is simplified as

$$\begin{aligned} &x^2(p^*x^2+q^*)\Phi''_{2m+1}(x) + x((r^*-2\theta p^*)x^2 + s^* - 2\theta q^*)\Phi'_{2m+1}(x) \\ &\quad + ((-2m(r^*+(2m-1)p^*) - \theta(r^*-(\theta+1)p^*)x^2 + \theta((\theta+1)q^*-s^*))\Phi_{2m+1}(x) = 0. \end{aligned} \tag{19}$$

If (19) is compared with the special case of (14) for $n=2m+1$, i.e.,

$$x^2(px^2+q)\Phi''_{2m+1}(x) + x(rx^2+s)\Phi'_{2m+1}(x) - (\alpha_{2m+1}x^2+\beta)\Phi_{2m+1}(x) = 0, \tag{20}$$

then equating two equations (19) and (20) yields

$$p^* = p, \quad q^* = q, \quad r^* = r + 2\theta p \quad \text{and} \quad s^* = s + 2\theta q. \tag{21}$$

Moreover the values $-\alpha_{2m+1}$ and $-\beta$ in (20) will be

$$\begin{aligned} -\alpha_{2m+1} &= -2m(r+2\theta p+(2m-1)p) - \theta(r+2\theta p-(\theta+1)p) \\ &= -(\theta+2m)(r+(\theta+2m-1)p), \\ -\beta &= \theta((\theta+1)q-(s+2\theta q)) = -\theta(s+(\theta-1)q). \end{aligned} \tag{22}$$

All these presented results finally lead to the following corollary.

Corollary 1. *The symmetric sequence*

$$\begin{aligned} \Phi_n(x) &= S_n^{(\theta)}\left(\begin{array}{cc|c} r & s & \\ p & q & x \end{array}\right) \\ &= x^{((1-(-1)^n)/2)\theta}\overline{S}_{2[n/2]}\left(\begin{array}{cc|c} r+(1-(-1)^n)\theta p, & s+(1-(-1)^n)\theta q & \\ p, & q & x \end{array}\right), \end{aligned} \tag{23}$$

which is equivalent to the explicit definition

$$S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)$$
$$= \prod_{j=0}^{[n/2]-1} \frac{(2j+1+(1-(-1)^n)\theta)q+s}{(2j-1+n+(1-(-1)^n)(\theta-1/2))p+r} x^{((1-(-1)^n)/2)(\theta-1)}$$
$$\times \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{j=0}^{[n/2]-(k+1)} \frac{(2j-1+n+(1-(-1)^n)(\theta-1/2))p+r}{(2j+1+(1-(-1)^n)\theta)q+s} \right) x^{n-2k} \tag{23.1}$$

satisfies

$$x^2(px^2+q)\Phi_n''(x)+x(rx^2+s)\Phi_n'(x)-\left(\alpha_n x^2+\beta\frac{1-(-1)^n}{2}\right)\Phi_n(x)=0, \tag{24}$$

where

$$\alpha_n=\left(n+(\theta-1)\frac{1-(-1)^n}{2}\right)\left(r+\left(n-1+(\theta-1)\frac{1-(-1)^n}{2}\right)p\right) \quad \text{and}$$
$$\beta=\theta(s+(\theta-1)q). \tag{24.1}$$

The explicit definition (23.1) helps us obtain a key identity in terms of polynomials (8.2) as

$$S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)$$
$$= x^{((1-(-1)^n)/2)(\theta-1)} \overline{S}_n\left(\begin{matrix} r+(1-(-1)^n)(\theta-1)p, & s+(1-(-1)^n)(\theta-1)q \\ p, & q \end{matrix}\middle| x\right), \tag{25}$$

which can be proved directly by (11). This identity is a useful tool to compute the recurrence relation of the defined functions $S_n^{(\theta)}(x;\, p,q,r,s)$. To reach this purpose, suppose

$$\overline{S}_n\left(\begin{matrix} r+(1-(-1)^n)(\theta-1)p, & s+(1-(-1)^n)(\theta-1)q \\ p, & q \end{matrix}\middle| x\right) = \overline{Q}_n(x).$$

Since $\overline{Q}_n(x)$ is a *monic symmetric polynomial*, by applying Maple software and some further computations in hand, the following three-term recurrence relation will be derived eventually:

$$\overline{Q}_{n+1}(x)=x\overline{Q}_n(x)+C_n^{(\theta)}\begin{pmatrix} r & s \\ p & q \end{pmatrix}\overline{Q}_{n-1}(x); \quad \overline{Q}_0(x)=1,\ \ \overline{Q}_1(x)=x, \tag{26}$$

where

$$C_n^{(\theta)}\begin{pmatrix} r & s \\ p & q \end{pmatrix}$$
$$= \frac{(1+(-1)^n(\theta-1))pqn^2+((\theta-3+3(-1)^n(1-\theta))pq+(1+(-1)^n(\theta-1))qr-(-1)^n\theta ps)n+((1-\theta)(p-r)q+((\theta-3)p+r)s)(1-(-1)^n)/2}{(((-1)^n(\theta-1)+2n+\theta-2)p+r)(((-1)^n(1-\theta)+2n+\theta-4)p+r)}. \tag{26.1}$$

Now it is sufficient to substitute $\overline{Q}_n(x)=x^{-((1-(-1)^n)/2)(\theta-1)}S_n^{(\theta)}(x;\, p,q,r,s)$ into (26) to obtain

$$S_{n+1}^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=\left(x^{1+(-1)^n(\theta-1)}\right)S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)+C_n^{(\theta)}\begin{pmatrix} r & s \\ p & q \end{pmatrix}S_{n-1}^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right). \tag{27}$$

For instance, if $n = 0, \ldots, 5$ in (27) and subsequently (26.1) then we have

$$S_0^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = 1, \quad S_1^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^{\theta},$$

$$S_2^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^2 + \frac{q+s}{p+r},$$

$$S_3^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^{\theta}\left(x^2 + \frac{(2\theta+1)q+s}{(2\theta+1)p+r}\right),$$

$$S_4^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^4 + 2\frac{3q+s}{5p+r}x^2 + \frac{(3q+s)(q+s)}{(5p+r)(3p+r)},$$

$$S_5^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^{\theta}\left(x^4 + 2\frac{(2\theta+3)q+s}{(2\theta+5)p+r}x^2 + \frac{((2\theta+3)q+s)((2\theta+1)q+s)}{((2\theta+5)p+r)((2\theta+3)p+r)}\right). \tag{28}$$

As items (28) show, $S_{2n}^{(\theta)}(x;\, p, q, r, s)$ are independent of θ. Therefore, by noting that $S_n^{(\theta)}(-x;\, p, q, r, s) = (-1)^n S_n^{(\theta)}$ $(x;\, p, q, r, s)$, the condition $(-1)^{\theta} = -1$ should be satisfied only for odd n. Moreover, for $\theta = 1$ in (23.1) and (26.1) we, respectively, have

$$S_n^{(1)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \overline{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \quad \text{and} \quad C_n^{(1)}\begin{pmatrix} r & s \\ p & q \end{pmatrix} = C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix}. \tag{29}$$

We can now obtain a generic orthogonality relation for the symmetric functions $S_n^{(\theta)}(x;\, p, q, r, s)$. As Eq. (24) and relation (4.1) of Theorem 1.1 show, the weight function corresponding to BCSOF is the same $W(x;\, p, q, r, s)$ as defined in (10) (i.e., independent of θ). So, according to Theorem 1.1, we should have

$$\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) S_m^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \mathrm{d}x = N_n \delta_{n,m}, \tag{30}$$

where

$$N_n = \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \left(S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)\right)^2 \mathrm{d}x \tag{30.1}$$

and $(px^2 + q)W(p, q, r, s; x)$ vanishes at $x = \alpha$ again. To compute the norm square value we can directly use the orthogonality relation (9). For this purpose, for $n = 2m$ we have

$$\begin{aligned} N_{2m} &= \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \left(S_{2m}^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)\right)^2 \mathrm{d}x \\ &= \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \left(\overline{S}_{2m}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)\right)^2 \mathrm{d}x \\ &= \left(\prod_{i=1}^{2m} C_i\begin{pmatrix} r & s \\ p & q \end{pmatrix}\right) \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \mathrm{d}x, \end{aligned} \tag{31}$$

while for $n = 2m + 1$ we have

$$\begin{aligned} N_{2m+1} &= \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \left(S_{2m+1}^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)\right)^2 \mathrm{d}x \\ &= \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) x^{2\theta}\left(\overline{S}_{2m}\left(\begin{matrix} r+2\theta p, & s+2\theta q \\ p, & q \end{matrix}\middle| x\right)\right)^2 \mathrm{d}x. \end{aligned} \tag{32}$$

On the other hand since

$$
\begin{aligned}
x^{2\theta} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) &= \exp\left(\int \frac{2\theta}{x}\,\mathrm{d}x\right)\exp\left(\int \frac{(r-2p)x^2+s}{x(px^2+q)}\,\mathrm{d}x\right) \\
&= \exp\left(\int \frac{(r+2\theta p-2p)x^2+s+2\theta q}{x(px^2+q)}\,\mathrm{d}x\right) \\
&= W\left(\begin{matrix} r+2\theta p, & s+2\theta q \\ p, & q \end{matrix}\middle|\, x\right),
\end{aligned} \tag{32.1}
$$

N_{2m+1}, by noting (31), is simplified as

$$
\begin{aligned}
N_{2m+1} &= \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r+2\theta p, & s+2\theta q \\ p, & q \end{matrix}\middle|\, x\right)\left(\overline{S}_{2m}\left(\begin{matrix} r+2\theta p, & s+2\theta q \\ p, & q \end{matrix}\middle|\, x\right)\right)^2 \mathrm{d}x \\
&= \left(\prod_{i=1}^{2m} C_i\left(\begin{matrix} r+2\theta p, & s+2\theta q \\ p, & q \end{matrix}\right)\right)\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r+2\theta p, & s+2\theta q \\ p, & q \end{matrix}\middle|\, x\right)\mathrm{d}x.
\end{aligned} \tag{33}
$$

Consequently, combining two relations (31) and (33) gives

$$
\begin{aligned}
N_n = &\prod_{i=1}^{2[n/2]} C_i\left(\begin{matrix} r+(1-(-1)^n)\theta p, & s+(1-(-1)^n)\theta q \\ p, & q \end{matrix}\right) \\
&\times \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r+(1-(-1)^n)\theta p, & s+(1-(-1)^n)\theta q \\ p, & q \end{matrix}\middle|\, x\right)\mathrm{d}x.
\end{aligned} \tag{34}
$$

The other standard properties of orthogonal functions (23.1) such as generating function, integral representation, hypergeometric representation and so on can directly be obtained by using (25) and referring to [7]. For instance, we proved in [7] that

$$
\begin{aligned}
S_n^{(1)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) &= \overline{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) \\
&= x^n\, {}_2F_1\left(\begin{matrix} -[n/2], & (q-s)/2q-[(n+1)/2] \\ & -(r+(2n-3)p)/2p \end{matrix}\middle|\, -\frac{q}{px^2}\right),
\end{aligned} \tag{35}
$$

where

$$
{}_2F_1\left(\begin{matrix} \alpha & \beta \\ & \gamma \end{matrix}\middle|\, x\right) = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}
$$

denotes a hypergeometric function of order (2,1) [5,9]. So, if one refers to (25), one eventually gets

$$
\begin{aligned}
S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) &= x^{(n+(1-(-1)^n)(\theta-1)/2)} \\
&\times {}_2F_1\left(\begin{matrix} -[n/2], & \dfrac{2-\theta+(-1)^n(\theta-1)}{2}-\dfrac{s}{2q}-[(n+1)/2] \\ & \dfrac{4-\theta-2n+(-1)^n(\theta-1)}{2}-\dfrac{r}{2p} \end{matrix}\middle|\, -\frac{q}{px^2}\right).
\end{aligned} \tag{36}
$$

Because of above-mentioned reasons, let us release other properties of BCSOF and in turn study four special cases of orthogonal functions (23.1).

3. Four main sub-classes of $S_n^{(\theta)}(x; p, q, r, s)$

First, we should note that the positive function (10), as an analogue of Pearson distributions family, satisfies the first-order differential equation

$$x\frac{\mathrm{d}}{\mathrm{d}x}((px^2+q)W(x)) = (rx^2+s)\,W(x), \tag{37}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2(px^2+q)W(x)) = x(r_1x^2+s_1)W(x) \quad \text{s.t.} \begin{cases} r_1 = r+2p, \\ s_1 = s+2q. \end{cases} \tag{37.1}$$

It is deduced from above equations that $W(p, q, r, s; x)$ is an analytic integrable function and its probability density function (pdf) is available.

In general, there are four main sub-classes of distributions family (10) (and consequently subsolutions of Eq. (37)) whose explicit pdfs are, respectively, as follows:

$$K_1 W\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right) = \frac{\Gamma(a+b+3/2)}{\Gamma(a+1/2)\Gamma(b+1)} x^{2a}(1-x^2)^b; \quad -1 \leqslant x \leqslant 1;$$
$$a+1/2>0; \quad b+1>0. \tag{38}$$

$$K_2 W\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right) = \frac{1}{\Gamma(a+1/2)} x^{2a}\exp(-x^2); \quad -\infty < x < \infty; \;\; a+1/2>0. \tag{39}$$

$$K_3 W\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right) = \frac{\Gamma(b)}{\Gamma(b+a-1/2)\Gamma(-a+1/2)} \frac{x^{-2a}}{(1+x^2)^b}; \quad -\infty < x < \infty;$$
$$b>0; \;\; a<1/2; \;\; b+a>1/2. \tag{40}$$

$$K_4 W\left(\begin{matrix} -2a+2, & 2 \\ 1, & 0 \end{matrix}\middle| x\right) = \frac{1}{\Gamma(a-1/2)} x^{-2a}\exp\left(-\frac{1}{x^2}\right); \quad -\infty < x < \infty; \;\; a>1/2. \tag{41}$$

The values $K_i; i = 1, 2, 3, 4$ in these distributions are the normalizing constants and $\Gamma(a) = \int_0^\infty x^{a-1}\mathrm{e}^{-x}\,\mathrm{d}x$ indicates the well-known gamma function [1]. Moreover, in each mentioned distributions the value of function vanishes at $x=0$, i.e. $W(p, q, r, s; 0) = 0$ for $s \neq 0$. The pdfs (38)–(41) play the role of a weight function for four symmetric orthogonal sequences introduced in Sections (3.1)–(3.4).

3.1. First subclass, a generalization of generalized ultraspherical polynomials

The generalized ultraspherical polynomials were first investigated by Chihara [4]. He obtained standard properties of these polynomials via a direct relation between them and Jacobi orthogonal polynomials. See also [2] for a further generalization of ultraspherical polynomials. According to (38), the characteristic vector corresponding to generalized ultraspherical polynomials is $(p, q, r, s) = (-1, 1, -2a-2b-2, 2a)$. So, if this vector is replaced in differential equation (6), then

$$x^2(1-x^2)\Phi_n''(x) - 2x((a+b+1)x^2-a)\Phi_n'(x)$$
$$+ (n(2a+2b+n+1)x^2 + ((-1)^n-1)a)\Phi_n(x) = 0 \tag{42}$$

has a basic solution as

$$\Phi_n(x) = S_n^{(1)}\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right), \tag{43}$$

which satisfies the orthogonality relation

$$\begin{aligned}&\int_{-1}^{1} W\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right) S_n^{(1)}\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right)\\ &\quad\times S_m^{(1)}\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right) \mathrm{d}x\\ &\quad= \left((-1)^n \prod_{i=1}^{n} C_i\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\right)\int_{-1}^{1} W\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right)\mathrm{d}x\right)\delta_{n,m},\end{aligned} \tag{44}$$

where

$$\begin{aligned}\int_{-1}^{1} W\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right)\mathrm{d}x &= \int_{-1}^{1} x^{2a}(1-x^2)^b\,\mathrm{d}x\\ &= B\left(a+\frac{1}{2}, b+1\right) = \frac{\Gamma(a+1/2)\,\Gamma(b+1)}{\Gamma(a+b+3/2)},\end{aligned} \tag{44.1}$$

and

$$C_n\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\right) = \frac{-(n+(1-(-1)^n)a)(n+(1-(-1)^n)a+2b)}{(2n+2a+2b-1)(2n+2a+2b+1)}. \tag{44.2}$$

From (44.1) one can conclude that $a+\frac{1}{2}>0$, $(-1)^{2a}=1$ and $b+1>0$. Moreover, $B(\lambda_1;\ \lambda_2)$ in this relation denotes the Beta integral [1,3] having various definitions

$$\begin{aligned}B(\lambda_1;\lambda_2) &= \int_0^1 x^{\lambda_1-1}(1-x)^{\lambda_2-1}\,\mathrm{d}x = \int_{-1}^{1} x^{2\lambda_1-1}(1-x^2)^{\lambda_2-1}\,\mathrm{d}x = \int_0^\infty \frac{x^{\lambda_1-1}}{(1+x)^{\lambda_1+\lambda_2}}\,\mathrm{d}x\\ &= 2\int_0^{\pi/2} \mathrm{Sin}^{(2\lambda_1-1)}x\,\mathrm{Cos}^{(2\lambda_2-1)}x\,\mathrm{d}x = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} = B(\lambda_2;\lambda_1).\end{aligned} \tag{45}$$

Now, to generalize the generalized ultraspherical polynomials with the same weight function and orthogonality interval $[-1, 1]$, it is enough to substitute the given characteristic vector into the main equation (24) to get

$$\begin{aligned}&x^2(1-x^2)\Phi_n''(x) - 2x((a+b+1)x^2-a)\Phi_n'(x)\\ &\quad+\left(\left(n+(\theta-1)\frac{1-(-1)^n}{2}\right)\left(n+2a+2b+1+(\theta-1)\frac{1-(-1)^n}{2}\right)x^2\right.\\ &\quad\left.-\theta(\theta+2a-1)\frac{1-(-1)^n}{2}\right)\Phi_n(x) = 0.\end{aligned} \tag{46}$$

It is clear that one of the basic solutions of this equation is

$$\Phi_n(x) = S_n^{(\theta)}\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right), \tag{46.1}$$

and according to (34), (8.1) and (44.1) have the orthogonality property

$$\int_{-1}^{1} x^{2a}(1-x^2)^b S_n^{(\theta)}\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right) S_m^{(\theta)}\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right) \mathrm{d}x$$

$$= \prod_{i=1}^{2[n/2]} \frac{(i+(1-(-1)^i)(a+(1-(-1)^n)\theta/2))(i+(1-(-1)^i)(a+(1-(-1)^n)\theta/2)+2b)}{(2i+2(a+(1-(-1)^n)\theta/2)+2b-1)(2i+2(a+(1-(-1)^n)\theta/2)+2b+1)}$$

$$\times B\left(a+\frac{1-(-1)^n}{2}\theta+\frac{1}{2}; b+1\right)\delta_{n,m}. \tag{47}$$

As (47) shows, the constraint on the parameters a, b and θ must be as $a+\frac{1}{2}>0$, $a+\theta+\frac{1}{2}>0$, $b+1>0$, $(-1)^{2a}=1$ and finally $(-1)^{\theta}=-1$.

An important note: Two important cases of functions (46.1) are when $a=0$, $b=-\frac{1}{2}$ or $a=0$, $b=\frac{1}{2}$ [in preprint]. Because by employing these cases one can appropriately generalize the *trigonometric type of Fourier expansions*. For instance, if $a=0$ and $b=-\frac{1}{2}$ in (46) then by the change of variable $x=\cos t$ one gets a generalization of differential equation of Fourier trigonometric sequences for $\theta=1$ as follows:

$$\Phi_n''(t)+\Bigg((n+(\theta-1)(1-(-1)^n)/2)^2$$

$$-\frac{\theta(\theta-1)}{\cos^2 t}\frac{1-(-1)^n}{2}\Bigg)\Phi_n(t)=0 \quad\Leftrightarrow\quad \theta>-\frac{1}{2}; \quad (-1)^{\theta}=-1. \tag{48}$$

3.2. Second subclass, a generalization of the generalized Hermite polynomials

The generalized Hermite polynomials were first introduced by Szego who presented a second order differential equation for them [10, Problem 25] almost as the same form as indicated in [7]. These polynomials can be characterized by a direct relationship between them and Laguerre orthogonal polynomials [9]. By referring to distribution (39), the characteristic vector corresponding to generalized Hermite polynomials is $(p, q, r, s)=(0, 1, -2, 2a)$. Hence, if this vector is replaced in (6) then the equation

$$x^2\Phi_n''(x)-2x(x^2-a)\Phi_n'(x)+(2nx^2+((-1)^n-1)a)\Phi_n(x)=0 \tag{49}$$

has a basic solution as

$$\Phi_n(x)=S_n^{(1)}\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right), \tag{49.1}$$

which satisfies

$$\int_{-\infty}^{\infty} x^{2a}\exp(-x^2) S_n^{(1)}\left(\begin{matrix} -2 & 2a \\ 0 & 1 \end{matrix}\middle| x\right) S_m^{(1)}\left(\begin{matrix} -2 & 2a \\ 0 & 1 \end{matrix}\middle| x\right) \mathrm{d}x$$

$$=\left(\frac{1}{2^n}\prod_{i=1}^{n}(1-(-1)^i)a+i\right)\Gamma\left(a+\frac{1}{2}\right)\delta_{n,m}. \tag{49.2}$$

Eq. (49.2) shows that the orthogonality relation is valid for $a+\frac{1}{2}>0$ and $(-1)^{2a}=1$. Now, similar to previous section, to generalize the generalized Hermite polynomials with the same weight function and also the same orthogonality interval, one should substitute the given characteristic vector into (24) to arrive at the equation

$$x^2\Phi_n''(x)-2x(x^2-a)\Phi_n'(x)+\Bigg(2\left(n+(\theta-1)\frac{1-(-1)^n}{2}\right)x^2$$

$$-\theta(2a+\theta-1)\frac{1-(-1)^n}{2}\Bigg)\Phi_n(x)=0. \tag{50}$$

According to expressed comments and noting (49.2), the basic solution of (50), i.e.,

$$\Phi_n(x) = S_n^{(\theta)}\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right) \tag{50.1}$$

has the orthogonality property

$$\begin{aligned}&\int_{-\infty}^{\infty} x^{2a}\exp(-x^2) S_n^{(\theta)}\left(\begin{matrix} -2 & 2a \\ 0 & 1 \end{matrix}\middle| x\right) S_m^{(\theta)}\left(\begin{matrix} -2 & 2a \\ 0 & 1 \end{matrix}\middle| x\right) \mathrm{d}x \\ &= \left(2^{-2[n/2]} \prod_{i=1}^{2[n/2]} i + (1-(-1)^i)\left(a + \frac{1-(-1)^n}{2}\theta\right)\right) \Gamma\left(a + \frac{1-(-1)^n}{2}\theta + \frac{1}{2}\right)\delta_{n,m}\end{aligned} \tag{51}$$

provided that $a + \frac{1}{2} > 0$, $a + \theta + \frac{1}{2} > 0$, $(-1)^{2a} = 1$ and $(-1)^{\theta} = -1$.

3.3. *Third subclass, a finite class of symmetric orthogonal functions with weight function $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$*

First, let us recall that by using Favard theorem for finite orthogonal polynomials, two finite sequences of symmetric orthogonal polynomials were introduced in [7]. See also [6] in this regard. According to density function (40), the characteristic vector corresponding to the first finite sequence is

$$(p, q, r, s) = (1,\ 1, -2a-2b+2, -2a). \tag{52}$$

If this vector is substituted in (9) for $\alpha = \infty$ then

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b} \bar{S}_n\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right) \bar{S}_m\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right) \mathrm{d}x \\ &= \left((-1)^n \prod_{i=1}^{n} C_i\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\right)\right) \frac{\Gamma(b+a-1/2)\Gamma(-a+1/2)}{\Gamma(b)}\delta_{n,m},\end{aligned} \tag{53}$$

if and only if $-C_{n+1}(1, 1, -2a-2b+2, -2a) > 0$ where

$$C_n\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\right) = \frac{(n-(1-(-1)^n)a)(n-(1-(-1)^n)a-2b)}{(2n-2a-2b+1)(2n-2a-2b-1)}. \tag{53.1}$$

The mentioned condition is in fact a consequence of Favard theorem for finite cases, which eventually lead to the following conditions:

$$\begin{cases} -2a-2b+n+m+1 \leqslant 0, \\ a < \frac{1}{2}; \quad b > 0; \quad b + a - \frac{1}{2} > 0. \end{cases} \tag{54}$$

This means that the polynomial set $\{\bar{S}_n(1, 1, -2a-2b+2, -2a; x)\}_{n=0}^{n=N}$ is finitely orthogonal with respect to the weight function $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$ if and only if $a < \frac{1}{2}$; $b + a - \frac{1}{2} > 0$; $b > 0$ and $N \leqslant b + a - \frac{1}{2}$. Now, let us consider the generalized sequence

$$\Phi_n(x) = S_n^{(\theta)}\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right) \tag{55}$$

satisfying the following differential equation:

$$x^2(x^2+1)\Phi_n''(x) - 2x((a+b-1)x^2+a)\Phi_n'(x)$$
$$-\left(\left(n+(\theta-1)\frac{1-(-1)^n}{2}\right)\left(n-2a-2b+1+(\theta-1)\frac{1-(-1)^n}{2}\right)x^2\right.$$
$$\left.+\theta(-2a+\theta-1)\frac{1-(-1)^n}{2}\right)\Phi_n(x)=0. \tag{56}$$

According to (30) and (34), the sequence (55) must have an orthogonality property as

$$\int_{-\infty}^{\infty}\frac{x^{-2a}}{(1+x^2)^b}S_n^{(\theta)}\left(\begin{matrix}-2a-2b+2, & -2a\\ 1, & 1\end{matrix}\,\middle|\,x\right)S_m^{(\theta)}\left(\begin{matrix}-2a-2b+2, & -2a\\ 1, & 1\end{matrix}\,\middle|\,x\right)\mathrm{d}x$$
$$=\prod_{i=1}^{2[n/2]}C_i\left(\begin{matrix}-2a-2b+2+(1-(-1)^n)\theta, & -2a+(1-(-1)^n)\theta\\ 1, & 1\end{matrix}\right)$$
$$\times\frac{\Gamma(b+a-(1-(-1)^n)\theta/2-1/2)\Gamma(-a+(1-(-1)^n)\theta/2+1/2)}{\Gamma(b)}\delta_{n,m}. \tag{57}$$

But, here an important question is how to determine the constraint of parameters in (57). There is an interesting technique to solve this problem. Let us write Eq. (56) in a self-adjoint form. Then according to Theorem 1.1, the following term must vanish:

$$[x^{-2a}(1+x^2)^{-b+1}(\Phi_n'(x)\Phi_m(x)-\Phi_m'(x)\Phi_n(x))]_{-\infty}^{\infty}=0. \tag{58}$$

On the other hand, since $\Phi_n(x)$ defined in (55) is a function of degree at most $n+(\theta-1)(1-(-1)^n)/2$, so we have

$$\max\ \deg(\Phi_n'(x)\Phi_m(x)-\Phi_m'(x)\Phi_n(x))=n+m-1+(\theta-1)(2-(-1)^n-(-1)^m)/2. \tag{59}$$

From (58) and (59) it can be deduced that

$$-2a+2(-b+1)+n+m-1+(\theta-1)(2-(-1)^n-(-1)^m)/2\leqslant 0. \tag{60}$$

Furthermore, the right-hand side of (57) shows that

$$b+a-\tfrac{1}{2}>0;\quad b+a-\tfrac{1}{2}-\theta>0;\quad -a+\tfrac{1}{2}>0;\quad -a+\tfrac{1}{2}+\theta>0\quad\text{and}\quad b>0, \tag{61}$$

which are equivalent to

$$b+a-\tfrac{1}{2}>0;\quad a<\tfrac{1}{2};\quad b>0\quad\text{and}\quad a-\tfrac{1}{2}<\theta<b+a-\tfrac{1}{2}. \tag{62}$$

Now we come back to (60). We have

$$-2a-2b+n+m+1+(\theta-1)(2-(-1)^n-(-1)^m)/2\leqslant 0. \tag{63}$$

In general, four cases may occur for n and m in inequality (63). They are, respectively,

$$\text{(i)}\begin{cases}n=2i,\\ m=2j+1,\end{cases}\quad\text{(ii)}\begin{cases}n=2i+1,\\ m=2j,\end{cases}\quad\text{(iii)}\begin{cases}n=2i,\\ m=2j,\end{cases}\quad\text{(iv)}\begin{cases}n=2i+1,\\ m=2j+1.\end{cases} \tag{64}$$

If each of the above cases is replaced in (63), by taking $N=\max\{m,n\}$ we get

$$\begin{cases}N\leqslant b+a-\theta/2 & \text{for the first and second cases of (64)},\\ N\leqslant b+a-\frac{1}{2} & \text{for the third case of (64)},\\ N\leqslant b+a-\theta+\frac{1}{2} & \text{for the fourth case of (64)}.\end{cases} \tag{65}$$

Finally assuming $\min\{b+a-\theta/2;\, b+a-\frac{1}{2};\, b+a-\theta+\frac{1}{2}\}=M_\theta^{(b+a)}$ gives the following corollary.

Corollary 2. *The finite set of symmetric functions* $\{S_n^{(\theta)}(1, 1, -2a-2b+2, -2a; x)\}_{n=0}^{n=N}$ *is orthogonal with respect to the weight function* $x^{-2a}(1+x^2)^{-b}$ *on* $(-\infty, \infty)$ *if and only if* $N \leqslant M_\theta^{(b+a)}$; $a < \frac{1}{2}$; $b > 0$; $b + a - \frac{1}{2} > 0$; $a - \frac{1}{2} < \theta < b + a - \frac{1}{2}$; $(-1)^{2a} = 1$ *and* $(-1)^\theta = -1$.

For example, suppose the even weight function $x^{-2/3}(1+x^2)^{-10}$ is given on $(-\infty, \infty)$. Hence we have $a = \frac{1}{3} < \frac{1}{2}$; $b = 10 > 0$ and $-\frac{1}{6} < \theta < \frac{59}{6}$. Consequently $M_\theta^{(31/3)} = \min\{\frac{62}{6} - \theta/2; \frac{61}{6}; \frac{63}{6} - \theta\}$ for any $-\frac{1}{6} < \theta < \frac{59}{6}$ and $(-1)^\theta = -1$. In other words, the finite functions set $\{S_n^{(\theta)}(1, 1, -\frac{56}{3}, -\frac{2}{3}; x)\}_{n=0}^{N \leqslant M_\theta^{(31/3)}}$ is orthogonal with respect to the weight function $x^{-2/3}(1+x^2)^{-10}$ on $(-\infty, \infty)$. For instance, let $\theta = \frac{1}{3} \in (-\frac{1}{6}, \frac{59}{6})$. In this case $(-1)^{1/3} = -1$; $M_{1/3}^{(31/3)} = \min\{\frac{61}{6}; \frac{61}{6}; \frac{61}{6}\}$ and $N = 10$, which eventually yields

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt[3]{x^2}(1+x^2)^{10}} S_n^{(1/3)}\left(\begin{matrix} -\frac{56}{3}, & \frac{-2}{3} \\ 1, & 1 \end{matrix}\middle| x\right) S_m^{(1/3)}\left(\begin{matrix} -\frac{56}{3}, & \frac{-2}{3} \\ 1, & 1 \end{matrix}\middle| x\right) \mathrm{d}x$$
$$= \prod_{i=1}^{2[n/2]} C_i \left(\begin{matrix} -(55+(-1)^n)/3, & -(1+(-1)^n)/3 \\ 1, & 1 \end{matrix}\right)$$
$$\times \frac{\Gamma((58+(-1)^n)/6)\Gamma((2-(-1)^n)/6)}{9!} \delta_{n,m} \quad \Leftrightarrow \quad m, \quad n \leqslant 10.$$

3.4. Fourth subclass, a finite class of symmetric orthogonal functions with weight function $x^{-2a}\exp(-1/x^2)$ *on* $(-\infty, \infty)$

By noting (41) if the characteristic vector

$$(p, q, r, s) = (1, 0, -2a+2, 2) \tag{66}$$

is replaced in (9) for $\alpha = \infty$ then

$$\int_{-\infty}^{\infty} x^{-2a} \exp\left(-\frac{1}{x^2}\right) \overline{S}_n \left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\middle| x\right) \overline{S}_m \left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\middle| x\right) \mathrm{d}x$$
$$= \left((-1)^n \prod_{i=1}^{n} C_i \left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\right)\right) \Gamma\left(a - \frac{1}{2}\right) \delta_{n,m}, \tag{67}$$

provided that

$$\left[x^{2-2a} \exp\left(-\frac{1}{x^2}\right) (\Phi_n'(x)\Phi_m(x) - \Phi_m'(x)\Phi_n(x))\right]_{-\infty}^{\infty} = 0, \tag{68}$$

or equivalently

$$2 - 2a + n + m - 1 \leqslant 0 \quad \Leftrightarrow \quad N \leqslant a - \tfrac{1}{2}; \quad N = \max\{m, n\}. \tag{69}$$

This means that the polynomial set $\{\overline{S}_n(1, 0, -2a+2, 2; x)\}_{n=0}^{n=N}$ is finitely orthogonal with respect to the weight function $x^{-2a}\exp(-1/x^2)$ on $(-\infty, \infty)$ if and only if $N \leqslant a - \frac{1}{2}$ and $(-1)^{2a} = 1$.

To extend this corollary, we can consider the generalized functions

$$\Phi_n(x) = S_n^{(\theta)}\left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\middle| x\right), \tag{70}$$

which satisfy the differential equation

$$x^4\Phi_n''(x)+2x((1-a)x^2+1)\Phi_n'(x)-\left(\left(n+(\theta-1)\frac{1-(-1)^n}{2}\right)\right.$$
$$\left.\times\left(n+1-2a+(\theta-1)\frac{1-(-1)^n}{2}\right)x^2+\theta\frac{1-(-1)^n}{2}\right)\Phi_n(x)=0. \tag{71}$$

According to (30) and (34), the sequence (70) should satisfy an orthogonality relation as

$$\int_{-\infty}^{\infty} x^{-2a}\exp\left(-\frac{1}{x^2}\right) S_n^{(\theta)}\left(\begin{matrix}-2a+2, & 2\\ 1, & 0\end{matrix}\middle| x\right) S_n^{(\theta)}\left(\begin{matrix}-2a+2, & 2\\ 1, & 0\end{matrix}\middle| x\right)\mathrm{d}x$$
$$=\prod_{i=1}^{2[n/2]} C_i\left(\begin{matrix}-2a+2+(1-(-1)^n)\,\theta, & 2\\ 1, & 0\end{matrix}\right)\Gamma\left(a-\frac{1}{2}-\frac{(1-(-1)^n)}{2}\theta\right)\delta_{n,m}. \tag{72}$$

Again, to determine parameters constraint in (72) we should apply the described technique. For this purpose, noting (68) and then (59) yields

$$-2a+n+m+1+(\theta-1)(2-(-1)^n-(-1)^m)/2\leqslant 0, \tag{73}$$

which is the same condition as (63) for $b=0$. Therefore, by referring to (64) and (65), if one defines $M_\theta^{(a)}=\min\{a-\theta/2;\ a-\frac{1}{2};\ a-\theta+\frac{1}{2}\}$ then the following corollary is derived.

Corollary 3. *The finite set of symmetric functions* $\{S_n^{(\theta)}(1,0,-2a+2,2;x)\}_{n=0}^{n=N}$ *is orthogonal with respect to the weight function* $x^{-2a}\exp(-1/x^2)$ *on* $(-\infty,\infty)$ *if and only if* $N\leqslant M_\theta^{(a)}$; $a-\frac{1}{2}>0$; $a-\theta-\frac{1}{2}>0$; $(-1)^{2a}=1$ *and finally* $(-1)^\theta=-1$.

4. A unified approach for the classification of BCSOF

First, it should be noted that the orthogonality interval of each introduced sub-classes, other than first one, is $(-\infty,\infty)$. So, applying a linear change of variable, say $x=wt+v$, preserves the orthogonality interval. Based on this note, we presented a unified approach in [7] with two special cases, i.e., a unified classification for given weight functions and a unified classification for given three-term recurrence relations of BCSOP. However, since the kind of recurrence relation of BCSOF is rather different from BCSOP, let us study it in this section. For this purpose, we should obtain the characteristic vector (p,q,r,s,θ) corresponding to a given three-term recurrence relation.

4.1. How to find the parameters p,q,r,s and θ if a special case of the main three-term recurrence equation (27) is given

In general, there are two ways to obtain the solution of a given recurrence equation of type (27). First way is to directly compare the given recurrence equation with (27). This method leads to a system of polynomial equations in terms of five parameters p,q,r,s and θ. Second way is to equate the first five terms of two recurrence equations together, which leads to a polynomial system with five equations and five unknowns p,q,r,s and θ, respectively. The following example clarifies this approach.

Example. Suppose the recurrence equation

$$U_{n+1}(x)=x^{1+2(-1)^n/5}U_n(x)-\frac{(70n-440)(-1)^n+440}{(10n-83+2(-1)^n)(10n-93-2(-1)^n)}U_{n-1}(x)$$

with initial data $U_0(x)=1$, $U_1(x)=x^{7/5}$ is given. Find its explicit solution, differential equation of solution, the related weight function and finally orthogonality property of solution.

Solution: If the above recurrence equation is directly compared with the main equation (27) and subsequently (26.1) then the values $(p, q, r, s, \theta) = (1, 0, -16, 2, \frac{7}{5})$ are, respectively, derived. Thus, the explicit solution of equation is the symmetric functions $S_n^{(7/5)}(1, 0, -16, 2; x)$, which satisfy the equation

$$x^4\Phi_n''(x) + x(-16x^2+2)\Phi_n'(x)$$
$$-((n+(1-(-1)^n)/5)(n-17+(1-(-1)^n)/5)x^2 + 7(1-(-1)^n)/5)\Phi_n(x) = 0.$$

Moreover, by replacing the characteristic vector in the generic weight function (10) as

$$W\left(\begin{matrix}-16 & 2\\ 1 & 0\end{matrix}\middle| x\right) = \exp\left(\int \frac{-18x^2+2}{x^3}\,\mathrm{d}x\right) = x^{-18}\mathrm{e}^{-1/x^2},$$

one can find out that the obtained solution is a particular case of the fourth introduced sub-class with $a = 9$. All these results lead to orthogonality relation

$$\int_{-\infty}^{\infty} x^{-18}\exp\left(-\frac{1}{x^2}\right) S_n^{(7/5)}\left(\begin{matrix}-16 & 2\\ 1 & 0\end{matrix}\middle| x\right) S_m^{(7/5)}\left(\begin{matrix}-16 & 2\\ 1 & 0\end{matrix}\middle| x\right)\mathrm{d}x$$
$$= \prod_{i=1}^{2[n/2]} C_i\left(\begin{matrix}-16+7(1-(-1)^n)/5, & 2\\ 1, & 0\end{matrix}\right)\Gamma\left(9 - \frac{7}{10}(1-(-1)^n) - \frac{1}{2}\right)\delta_{n,m}.$$

Note that this orthogonality property is valid only for $m, n \leqslant 8$, because we have $M_{7/5}^{(9)} = \min\{\frac{83}{10}; \frac{85}{10}; \frac{81}{10}\}$ and therefore $N = 8$.

5. An important remark

In the previous work [7], we pointed out that differential equation (6) could be extended to a generic operator equation of the form

$$x^2(px^2+q)\,\mathbf{L}^2 P_n(x) + x(rx^2+s)\,\mathbf{L}P_n(x)$$
$$-(n(r+(n-1)p)x^2 + (1-(-1)^n)s/2)P_n(x) = 0, \tag{74}$$

where the linear operator $\mathbf{L}$, known as Hahn's operator [5, p. 159], is defined by

$$\mathbf{L}(f(x)) = \frac{f(vx+w)-f(x)}{(v-1)x+w}; \quad v, w \in \mathbf{R}. \tag{75}$$

Now, here we would like to add that the operator equation (74) could be extended as

$$x^2(px^2+q)\mathbf{L}^2\Phi_n(x) + x(rx^2+s)\mathbf{L}\Phi_n(x)$$
$$-((n+(\theta-1)(1-(-1)^n)/2))(r+(n-1+(\theta-1)(1-(-1)^n)/2)p)x^2$$
$$+\theta(s+(\theta-1)q)(1-(-1)^n)/2))\Phi_n(x) = 0. \tag{76}$$

It is clear that the ordinary derivative operator in (76), i.e., when $w = 0$ and $v \to 1$, corresponds to Eq. (24). Hence, two further cases, i.e., the difference operator Δ for $v = w = 1$ (or equivalently ∇ for $v = -w = 1$) and the v-difference operator D_v for $w = 0$ remain for general equation (76) that should be studied and investigated separately.

Finally let us add that the extension of Sturm–Liouville problems for discrete variables is possible and thus there is a basic class of discrete orthogonal functions that satisfies a generic difference equation and has four special orthogonal subclasses [in preparation].

6. Conclusion

In this research, by using the extended Sturm–Liouville theorem for symmetric functions, we defined a generic second-order differential equation and then a basic class of symmetric orthogonal functions with five parameters.

This class satisfied a generic orthogonality relation whose weight function corresponds to an analogue of Pearson distributions. The following table summarizes the primary properties of four introduced subclasses of BCSOF.

Four special sub-cases of $S_n^{(\theta)}(p, q, r, s; x)$

Definition	Weight function	Interval and kind
(1) $S_n^{(\theta)}\left(\begin{array}{cc\|c} -2a-2b-2, & 2a & x \\ -1, & 1 & \end{array}\right)$	$x^{2a}(1-x^2)^b$	$[-1, 1]$, Infinite
(2) $S_n^{(\theta)}\left(\begin{array}{cc\|c} -2, & 2a & x \\ 0, & 1 & \end{array}\right)$	$x^{2a}\exp(-x^2)$	$(-\infty, \infty)$, Infinite
(3) $S_n^{(\theta)}\left(\begin{array}{cc\|c} -2a-2b+2, & -2a & x \\ 1, & 1 & \end{array}\right)$	$\dfrac{x^{-2a}}{(1+x^2)^b}$	$(-\infty, \infty)$, Finite
(4) $S_n^{(\theta)}\left(\begin{array}{cc\|c} -2a+2, & 2 & x \\ 1, & 0 & \end{array}\right)$	$x^{-2a}\exp(-1/x^2)$	$(-\infty, \infty)$, Finite

Note in this table that in addition to parameters constraint the condition $(-1)^{2a} = 1$ is always necessary for each introduced weight functions. Hence, they can, respectively, be represented as $|x|^{2a}(1-x^2)^b$, $|x|^{2a}\exp(-x^2)$, $|x|^{-2a}(1+x^2)^{-b}$ and $|x|^{-2a}\exp(-1/x^2)$.

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